

# DIVERGENCE OF SPECTRAL DECOMPOSITIONS OF HILL OPERATORS WITH TWO EXPONENTIAL TERM POTENTIALS

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ABSTRACT. We consider the Hill operator

$$Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi,$$

subject to periodic or antiperiodic boundary conditions ( $bc$ ) with potentials of the form

$$v(x) = ae^{-2irx} + be^{2isx}, \quad a, b \neq 0, \quad r, s \in \mathbb{N}, \quad r \neq s.$$

It is shown that the system of root functions does not contain a basis in  $L^2([0, \pi], \mathbb{C})$  if  $bc$  are periodic or if  $bc$  are antiperiodic and  $r, s$  are odd or  $r = 1$  and  $s \geq 3$ .

*Keywords:* Hill operators, periodic and antiperiodic boundary conditions, two exponential term potentials

*MSC:* 47E05, 34L40, 34L10

## 1. INTRODUCTION

We consider the Hill operators  $L = L_{Per^\pm}(v)$  with smooth  $\pi$ -periodic (complex-valued) potentials  $v$

$$(1.1) \quad Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi,$$

subject to periodic ( $Per^+$ ) or antiperiodic ( $Per^-$ ) boundary conditions:

$$Per^\pm : \quad y(\pi) = \pm y(0), \quad y'(\pi) = \pm y'(0).$$

See basics and details in [15].

If  $v$  is real-valued, then  $L_{Per^\pm}(v)$  is a self-adjoint operator with a discrete spectrum. The system of its normalized eigenfunctions

$$(1.2) \quad \Phi = \{\varphi_k : L\varphi_k = \lambda_k \varphi_k, \quad \|\varphi_k\| = 1\}$$

is orthonormal, and the spectral decompositions

$$(1.3) \quad f = \sum_k \langle f, \varphi_k \rangle \varphi_k$$

converge (unconditionally) in  $L^2([0, \pi])$  for every  $f \in L^2([0, \pi])$ .

If  $v$  is a complex-valued potential the picture becomes more complicated – see [11, 12, 14, 18, 19, 20, 23, 24, 25]. In 2006 A. Makin [16, 17]

and the authors [3, Thm 71] gave the first examples of such potentials that the system of root functions for periodic or antiperiodic boundary conditions does not contain a basis in  $L^2([0, \pi])$  even though there are all but finitely many eigenvalues are simple.

It is well known that the spectra of the operators  $L_{Per\pm}$  are discrete, and the following localization formulas hold (see, for example, [4, Prop 1]):

$$(1.4) \quad Sp(L_{Per\pm}) \subset \Pi_N \cup \bigcup_{n>N, n \in \Gamma^\pm} D_n, \quad \#\{Sp(L_{Per\pm}) \cap D_n\} = 2,$$

where  $D_n = \{z : |z - n^2| < 1\}$ ,  $\Gamma^+ = 2\mathbb{N}$ ,  $\Gamma^- = 2\mathbb{N} - 1$ ,  $N = N(v)$ ,

$$(1.5) \quad \Pi_N = \{z = x + iy \in \mathbb{C} : |x| < (N + 1/2)^2, |y| < N\}.$$

In either case the spectral block decompositions

$$(1.6) \quad g = S_N g + \sum_{n>N, n \in \Gamma_\pm} P_n g, \quad \forall g \in L^2([0, \pi]),$$

where

$$(1.7) \quad S_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} (z - L_{Per\pm})^{-1} dz, \quad P_n = \frac{1}{2\pi i} \int_{\partial D_n} (z - L_{Per\pm})^{-1} dz,$$

converge unconditionally in  $L^2([0, \pi])$ . This is true even if the  $\pi$ -periodic potential  $v$  is singular, i.e.,  $v \in H_{loc}^{-1}(\mathbb{R})$ , as A. Savchuk and A. Shkalikov showed in [22]. An alternative proof is given in [5].

The unconditional convergence of decompositions (1.6) implies that for every set  $\Delta$  (finite or infinite) of even (or odd) integers  $n > N$  the sum of projections

$$(1.8) \quad P(\Delta) = \sum_{k \in \Delta} P_k$$

converges unconditionally, so the projections  $P(\Delta)$  are well defined and

$$(1.9) \quad \sup_{\Delta} \|P(\Delta)\| \leq M(v) < \infty.$$

Invariant subspaces  $E(\Delta) = \text{Ran } P(\Delta)$  have  $\{P_k, k \in \Delta\}$  as their Riesz system of projections,  $\dim P_k = 2$ .

Could  $P_k$  be split to give a basis of root functions for  $E(\Delta)$ ? We put the question in this way because for one and the same operator  $L_{Per\pm}(v)$  the answer could be *yes* and *no* depending on  $\Delta$ . For example, if

$$v(x) = ae^{-10ix} + be^{10ix}$$

and

$$(1.10) \quad \Delta_0 = \{n \in \Gamma^\pm : n \not\equiv 0 \pmod{5}\},$$

then the answer is positive, but for  $\Delta_1 = 5\mathbb{N}$  the answer is *no* if  $|a| \neq |b|$ , and *yes* if  $|a| = |b|$ . We explain this phenomenon in Section 4 (see Proposition 19).

In view of (1.8) and (1.9), the following holds (see Corollary 10 in [9, Section 3] for details).

**Remark 1.** *If  $\Delta$  is an infinite set of even (or odd) integers, then the corresponding system of periodic (or antiperiodic) root functions contains a basis of  $E(\Delta)$  if and only if it contains an unconditional basis of  $E(\Delta)$ .*

The spectra localization formula (1.4) allows us to apply the Lyapunov–Schmidt projection method (see [3, Lemma 21]) and reduce the eigenvalue equation  $Ly = \lambda y$  to a series of eigenvalue equations in two-dimensional eigenspaces  $E_n^0$  of the free operator. This leads to the following (see [3, Section 2.2]).

**Lemma 2.** *Let  $L$  be a Hill operator with a potential  $v \in L^2$ . Then, for large enough  $n \in \mathbb{N}$ , there are functionals  $\alpha_n(v; z)$  and  $\beta_n^\pm(v; z)$ ,  $|z| < n$  such that a number  $\lambda = n^2 + z$ ,  $|z| < n/4$ , is a periodic (for even  $n$ ) or anti-periodic (for odd  $n$ ) eigenvalue of  $L$  if and only if  $z$  is an eigenvalue of the matrix*

$$(1.11) \quad \begin{bmatrix} \alpha_n(v; z) & \beta_n^-(v; z) \\ \beta_n^+(v; z) & \alpha_n(v; z) \end{bmatrix}.$$

Moreover,  $\alpha_n(z; v)$  and  $\beta_n^\pm(z; v)$  depend analytically on  $v$  and  $z$ , and  $z_n^- = \lambda_n^- - n^2$  and  $z_n^+ = \lambda_n^+ - n^2$  are the only solutions of the equation

$$(1.12) \quad (z - \alpha_n(v; z))^2 = \beta_n^-(v; z)\beta_n^+(v; z).$$

The functionals  $\alpha_n(v; z)$  and  $\beta_n^\pm(v; z)$  are well defined for large enough  $n$  by explicit expressions in terms of the Fourier coefficients of the potential (see [3, Formulas (2.16)-(2.33)] for Hill operators with  $L^2$ -potentials).

Here we provide formulas for  $\alpha_n(v; z)$  and  $\beta_n^\pm(v; z)$  using the combinatorial approach that has been developed in [2, 4] and used there to obtain the asymptotics of the spectral gaps  $\gamma_n = \lambda_n^+ - \lambda_n^-$  for potentials of the form  $v(x) = a \cos 2x + b \cos 4x$ .

For each  $n \in \mathbb{N}$  a *walk*  $x$  from  $-n$  to  $n$  (or from  $n$  to  $-n$  or from  $n$  to  $n$ ) is defined through its *sequence of steps*

$$(1.13) \quad x = (x(t))_{t=1}^{\nu+1}, \quad 1 \leq \nu = \nu(x) < \infty,$$

where  $x(t) \in 2\mathbb{Z} \setminus \{0\}$ , and respectively,

$$(1.14) \quad \sum_{t=1}^{\nu+1} x(t) = 2n \quad \left( \text{or} \quad \sum_{t=1}^{\nu+1} x(t) = -2n \quad \text{or} \quad \sum_{t=1}^{\nu+1} x(t) = 0 \right).$$

A walk  $x$  is called *admissible* if its *vertices*  $j(t) = j(t, x)$  given, respectively, by

$$(1.15) \quad j(0) = -n \quad \text{or} \quad j(0) = +n$$

and

$$(1.16) \quad j(t) = -n + \sum_{i=1}^t x(i) \quad \text{or} \quad j(t) = n + \sum_{i=1}^t x(i), \quad 1 \leq t \leq \nu + 1,$$

satisfy

$$(1.17) \quad j(t) \neq \pm n \quad \text{for} \quad 1 \leq t \leq \nu.$$

Let

$$(1.18) \quad v = \sum_{m \in 2\mathbb{Z}} V(m) e^{imx}$$

be the Fourier expansion of the potential  $v$  with respect to the system  $\{e^{imx}, m \in 2\mathbb{Z}\}$ , and let  $X_n, Y_n$  and  $W_n$  be, respectively, the set of all admissible walks from  $-n$  to  $n$ , from  $n$  to  $-n$  and from  $n$  to  $n$ . For each admissible walk  $x$  we set

$$(1.19) \quad h_1(x; z) = \prod_{t=1}^{\nu} [n^2 - j(t)^2 + z]^{-1}, \quad h(x) = h_1(x) \prod_{t=1}^{\nu+1} V(x(t));$$

then

$$(1.20) \quad \alpha_n(z) = \sum_{x \in W_n} h(x, z), \quad \beta_n^+(z) = \sum_{x \in X_n} h(x, z), \quad \beta_n^-(z) = \sum_{x \in Y_n} h(x, z).$$

The core of our approach is analysis of asymptotic behavior of the functionals  $\beta_n^\pm(z) = \beta_n^\pm(v; z)$ . In particular, the following criterion (which is a slight modification of Theorem 1 in [7] or Theorem 2 in [6]) gives a constructive approach to determine the basisness properties of the root function system.

**Criterion 3.** *Let  $v \in L^2([0, \pi])$ , and let  $\Delta \subset \Gamma^+$  (or  $\Delta \subset \Gamma^-$ ) be an infinite set of sufficiently large numbers. If  $\Delta = \Delta_0 \cup \Delta_1$ , where*

$$(1.21) \quad \beta_n^+(z) \equiv \beta_n^-(z) \equiv 0 \quad \text{for } n \in \Delta_0,$$

$$(1.22) \quad \beta_n^+(0) \neq 0, \quad \beta_n^-(0) \neq 0 \quad \text{for } n \in \Delta_1$$

and there is a constant  $c > 0$  such that

$$(1.23) \quad c^{-1} |\beta_n^\pm(0)| \leq |\beta_n^\pm(z)| \leq c |\beta_n^\pm(0)|, \quad \text{for } n \in \Delta_1, \quad |z| \leq 1,$$

then:

(a) *for large enough  $n \in \Delta$ , the operator  $L_{Per^\pm}(v)$  has in the disc  $D_n = \{z : |z - n^2| < 1\}$  exactly one periodic (or antiperiodic) eigenvalue*

of geometric multiplicity 2 if  $n \in \Delta_0$ , and exactly two simple periodic (or antiperiodic) eigenvalues if  $n \in \Delta_1$ ;

(b) the system of root functions of  $L_{Per^\pm}(v)$  contains a Riesz basis of  $E(\Delta)$  if and only if

$$(1.24) \quad \limsup_{n \in \Delta_1} t_n(0) < \infty,$$

where

$$(1.25) \quad t_n(z) = \max\{|\beta_n^-(z)|/|\beta_n^+(z)|, |\beta_n^+(z)|/|\beta_n^-(z)|\}.$$

In the framework of this criterion one can explain practically all known cases of existence or non-existence of bases consisting of root functions of the operators  $L_{Per^\pm}(v)$  for specific classes of potentials  $v$ . For example, the main result in [26] follows from Criterion 3.

In general form, i.e., without the restrictions (1.21) - (1.23), Criterion 3 is given in [8] in the context of 1D Dirac operators but the formulation and proof are the same in the case of Schrödinger operators (see Proposition 19 in [9]). Moreover, the same argument gives the following more general statement.

**Criterion 4.** Let  $\Gamma^+ = 2\mathbb{N}$ ,  $\Gamma^- = 2\mathbb{N} - 1$  in the case of Hill operators with  $H_{per}^{-1}$ -potentials, and  $\Gamma^+ = 2\mathbb{Z}$ ,  $\Gamma^- = 2\mathbb{Z} - 1$  in the case of one dimensional Dirac operators with  $L^2$ -potentials. There exists  $N_* = N_*(v)$  such that for  $|n| > N_*$  the operator  $L = L_{Per^\pm}(v)$  has in the disc  $D_n = \{z : |z - n^2| < n/2\}$  (respectively  $D_n = \{z : |z - n| < 1/2\}$ ) exactly two periodic (for  $n \in \Gamma^+$ ) or antiperiodic (for  $n \in \Gamma^-$ ) eigenvalues, counted with multiplicity. Let

$$\mathcal{M}^\pm = \{n \in \Gamma^\pm : n \geq N_*, \lambda_n^- \neq \lambda_n^+\}.$$

(a) If  $\Delta \subset \Gamma^\pm$  is an infinite set such that  $|n| > N_*$  for  $n \in \Delta$ , then the system of periodic (or antiperiodic) root functions contains a Riesz basis in  $E(\Delta)$  if and only if

$$(1.26) \quad \limsup_{n \in \Delta \cap \mathcal{M}^\pm} t_n(z_n^*) < \infty,$$

where  $z_n^* = \frac{1}{2}(\lambda_n^- + \lambda_n^+) - \lambda_n^0$  with  $\lambda_n^0 = n^2$  for Hill operators and  $\lambda_n^0 = n$  for Dirac operators.

(b) The system of root functions of  $L_{Per^\pm}(v)$  contains a Riesz basis, (respectively, in  $L^2([0, \pi])$  in the Hill case or in  $L^2([0, \pi], \mathbb{C}^2)$  in the Dirac case) if and only if (1.26) holds for  $\Delta = \Gamma^\pm$ .

Another interesting abstract criterion of basisness is the following.

**Criterion 5.** *The system of root functions of the operator  $L_{Per^\pm}(v)$  contains a Riesz basis in  $E(\Delta)$  if and only if*

$$(1.27) \quad \limsup_{n \in \Delta \cap \mathcal{M}^\pm} \frac{|\lambda_n^+ - \mu_n|}{|\lambda_n^+ - \lambda_n^-|} < \infty,$$

where (for large enough  $n$ )  $\mu_n$  is the Dirichlet eigenvalue close to  $n^2$ .

In the case  $\Delta = \Gamma^\pm$  this criterion was given (with completely different proofs) in [13] for Hill operators with  $L^2$ -potentials and in [9] for Hill operators with  $H_{per}^{-1}$ -potentials and for one-dimensional Dirac operators with  $L^2$ -potentials as well. The proof of the criterion in the more general case  $\Delta \subset \Gamma^\pm$  is the same.

However, if one wants to apply Criterion 5 to specific potentials  $v$ , say  $v(x) = a \cos 2x + b \cos 4x$  with  $a, b \in \mathbb{C}$ , it is necessary first to obtain the asymptotics of the spectral gaps  $|\lambda_n^+ - \lambda_n^-|$  and deviations  $|\mu_n - \lambda_n^+|$ , what is by itself quite a difficult problem.

In [6, 7] we considered low degree trigonometric polynomials with nonzero coefficients  $v(x)$  of the form

- (i)  $ae^{-2ix} + be^{2ix}$ ;
- (ii)  $ae^{-2ix} + Be^{4ix}$ ;
- (iii)  $ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{4ix}$ .

It is shown that the system of eigenfunctions and (at most finitely many) associated functions is complete but it is not a basis in  $L^2([0, \pi], \mathbb{C})$  if  $|a| \neq |b|$  in the case (i), if  $|A| \neq |B|$  and neither  $-b^2/4B$  nor  $-a^2/4A$  is an integer square in the case (iii), and it is never a basis in the case (ii) subject to periodic boundary conditions. In connection with Example (iii) see also [1, 21].

In this paper we extend the analysis of the above example (ii) to potentials of the form

$$v(x) = ae^{-2irx} + be^{2isx}, \quad a, b \neq 0, \quad r, s \in \mathbb{N}, \quad r \neq s.$$

In Section 2, Theorem 11, it is shown that the system of root functions does not contain a basis in  $L^2([0, \pi], \mathbb{C})$  for periodic  $bc$  or if  $bc$  is antiperiodic but  $r, s$  are odd.

In Section 3, the case  $r = 1, s > 2$  any (i.e., odd or even) with antiperiodic boundary conditions is completely analyzed as well, and it is shown that the system of root functions does not contain a basis in  $L^2([0, \pi], \mathbb{C})$  – see Theorem 18.

In our proofs we face series of questions related to enumerative combinatorics and diophantine equations. Their solution would dramatically extend the class of trigonometric polynomial potentials  $v(x)$  for which the problem of convergence of spectral decompositions could be

resolved. In our study of potentials (iii) in [2, 4] we discover a combinatorial identity (see also [1, 21]) that could be a prototype of such results. In this connection see [10] for more comments and open problems.

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## 2. TWO EXPONENTIAL TERM POTENTIALS

1. Our main objects are the potentials of the form

$$(2.1) \quad v(x) = ae^{-2Rix} + be^{2Six}, \quad a, b \neq 0,$$

with  $R, S \in \mathbb{N}$ ,  $R \neq S$ . Then

$$(2.2) \quad R = dr, \quad S = ds, \quad \text{where } r, s \text{ are coprime;}$$

they are the main parameters in what follows.

In view of (2.1), an admissible path  $x = (x(t))_{t=1}^{\nu+1}$  from  $-n$  to  $n$  gives a non-zero term  $h(x, z)$  in  $\beta_n^+(z)$  (see (1.20)) if and only if

$$(2.3) \quad x(t) \in \{-2R, 2S\}, \quad t = 1, 2, \dots, \nu + 1.$$

Let  $x$  be such a path, and let

$$(2.4) \quad \tilde{p} = \#\{t : x(t) = -2R\}, \quad \tilde{q} = \#\{t : x(t) = 2S\}.$$

Consider

$$(2.5) \quad n \in \Delta := (rsd)\mathbb{N}, \quad \text{i.e., } n = rsdm, \quad m \in \mathbb{N};$$

then

$$(2.6) \quad -2R\tilde{p} + 2S\tilde{q} = 2n, \quad s\tilde{q} = r\tilde{p} + rsm.$$

and therefore,

$$(2.7) \quad \tilde{q} = rq, \quad \tilde{p} = sp \quad \text{with } q = p + m.$$

Under the assumptions (2.4) - (2.7) we denote by  $X_n(p)$  the set of all admissible paths from  $-n$  to  $n$  with  $\tilde{p} = ps$  negative steps  $-2R$  and  $\tilde{q} = qr$  positive steps  $2S$ . Then  $n \in \Delta$  (see (2.5)) implies

$$(2.8) \quad \#X_n(0) = 1, \quad X_n(0) = \{x^*\},$$

where

$$(2.9) \quad x^*(k) = 2sd, \quad j_k^* := j(k, x^*) = -n + 2sdk, \quad 1 \leq k \leq rm - 1.$$

Therefore, for  $n = rsdm$  we have  $n^2 - (j_k^*)^2 = 4s^2d^2k(rm - k)$ , which implies that

$$(2.10) \quad h(x^*, 0) = b^{rm} \prod_{k=1}^{rm-1} \frac{1}{n^2 - (j_k^*)^2} = \frac{b^{rm}}{(4s^2d^2)^{rm-1}[(rm-1)!]^2}.$$

Moreover, in these notations, we have

$$(2.11) \quad \beta_n^+(z) = \sum_{p=0}^{\infty} \sum_{x \in X_n(p)} h(x, z),$$

where, for  $x \in X_n(p)$ ,

$$(2.12) \quad h(x, z) = a^{\tilde{p}} b^{\tilde{q}} h_1(x, z), \quad h_1(x, z) = \prod_{t=1}^{\tilde{p}+\tilde{q}-1} (n^2 - j(t, x)^2 + z)^{-1}.$$

2. Next we show that the leading term in the asymptotics of  $\beta_n^+(z)$  is determined by  $h(x^*, 0)$  only. Fix  $p \geq 1$  and  $x \in X_n(p)$ ; choose a set of vertices  $j(t_k, x)$ ,  $k = 1, \dots, rm - 1$  so that

$$(2.13) \quad 0 \leq \delta_k := j_k^* - j(t_k, x) < 2S = 2sd.$$

(This is possible since the positive steps of  $x$  are equal to  $2S$ .)

We have  $h_1(x, z) = \Pi_1(z) \cdot \Pi_2(z)$ , where

$$\Pi_1(z) = \prod_{k=1}^{rm-1} (n^2 - j(t_k, x)^2 + z)^{-1}$$

and  $\Pi_2(z)$  is the product of those factors of  $h_1(x, z)$  which are not included in  $\Pi_1(z)$ . In view of (2.4) and (2.7), the number of factors in  $\Pi_2(z)$  is equal to

$$\nu(x) - (rm - 1) = \tilde{p} + \tilde{q} - 1 - (rm - 1) = (r + s)p.$$

For  $n \geq 2$  and  $|z| \leq 1$  we have

$$|n^2 - j(t_k, x)^2 + z| \geq |n^2 - j(t_k, x)^2| - 1 \geq 2n - 2 \geq n,$$

so the absolute value of each factor is less than  $1/n$ . Therefore,

$$(2.14) \quad |\Pi_2(z)| \leq (1/n)^{(r+s)p}.$$

To estimate  $\Pi_1(z)$  we need the following (compare with [7, Lemma 2]).

**Lemma 6.** *If  $\{j_1, \dots, j_K\} \subset \{j = -n + 2t, t = 1, \dots, n-1\}$ , then for large enough  $n$  and  $|z| \leq 1$*

$$(2.15) \quad \prod_{k=1}^K |n^2 - j_k^2 + z|^{-1} = \left( \prod_{k=1}^K |n^2 - j_k^2|^{-1} \right) (1 + \theta_n), \quad |\theta_n| \leq \frac{4 \log n}{n}.$$



*Proof.* Indeed, we have

$$\theta_n = \prod_{k=1}^K \frac{n^2 - (j_k)^2}{n^2 - (j_k)^2 + z} - 1 = e^{-w_n} - 1,$$

where  $w_n = \sum_{k=1}^K \log \left( 1 + \frac{z}{n^2 - (j_k)^2} \right)$ . Therefore, by the inequality

$$|\log(1 + \zeta)| \leq \sum_{k=1}^{\infty} |\zeta|^k \leq 2|\zeta| \quad \text{for } |\zeta| \leq 1/2,$$

it follows that for large enough  $n$

$$|w_n| \leq \sum_{k=1}^K \frac{2|z|}{n^2 - (j_k)^2} \leq \sum_{k=1}^{n-1} \frac{2}{n^2 - (-n + 2k)^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \leq \frac{2 \log n}{n} < \frac{1}{2}.$$

On the other hand, if  $|w| \leq 1/2$  then  $|e^{-w} - 1| \leq \sum_{k=1}^{\infty} |w|^k \leq 2|w|$ , which implies (2.15).  $\square$

Now we could estimate the product  $\Pi_1(z)$  by Lemma 6. Indeed, if  $j_k = j(t_k, x)$  then due to the choice of  $t_k$  (see (2.13)) the vertices  $j_k$  are distinct and  $-n < j_k < n$ . Therefore, (2.15) implies that

$$(2.16) \quad \Pi_1(z) = \Pi_1(0)(1 + \theta_n), \quad \text{where } |\theta_n| = O\left(\frac{\log n}{n}\right).$$

3. Next we estimate  $\Pi_1(0)$  by comparing it with  $h_1(x^*, 0)$ . To this end we need the following.

**Lemma 7.** *Let  $n, K, S \in \mathbb{N}$  and  $n \geq (K + 1)S$ , and let*

$$(2.17) \quad j_k = \pm(n - 2kS), \quad 0 \leq \delta_k \leq 2(S - d), \quad k = 1, \dots, K, \quad d \in (0, S).$$

*Then*

$$(2.18) \quad \prod_{k=1}^K \frac{n^2 - (j_k)^2}{n^2 - (j_k - \delta_k)^2} \leq Cn^{1-d/S}.$$

(This lemma is a more general assertion than Lemma 12 in [7], where  $S = 2$  and  $\delta_k = 2$  so  $d = 1$ .)

*Proof.* First we consider the case  $j_k = -(n - 2kS)$ , i.e., moving forward from  $-n$  to  $+n$ . Then  $n - j_k = 2n - 2kS \geq 2S$ , and we have

$$\frac{n^2 - (j_k)^2}{n^2 - (j_k - \delta_k)^2} = \frac{(n + j_k)(n - j_k)}{(n + j_k - \delta_k)(n - j_k + \delta_k)} \leq \frac{n + j_k}{n + j_k - \delta_k}.$$

If  $j_k = -n + 2Sk$ , then

$$\frac{n + j_k}{n + j_k - \delta_k} = \left(1 - \frac{\delta_k}{2kS}\right)^{-1} \leq \left(1 - \frac{S-d}{kS}\right)^{-1}.$$

Therefore, the product in (2.18) does not exceed

$$\prod_{k=1}^K \left(1 - \frac{\gamma}{k}\right)^{-1} \leq Cn^\gamma \quad \text{where } \gamma = 1 - \frac{d}{S}, \quad C = C(\gamma).$$

When we are moving backward from  $+n$  to  $-n$ , then  $j_k = n - 2Sk$ , so

$$\frac{n + j_k}{n + j_k - \delta_k} = \left(1 - \frac{\delta_k}{2n - 2kS}\right)^{-1} \leq \left(1 - \frac{(S-d)}{(K+1-k)S}\right)^{-1}.$$

Therefore, the product in (2.18) does not exceed

$$\prod_{k=1}^K \left(1 - \frac{\gamma}{K+1-k}\right)^{-1} \leq Cn^\gamma \quad \text{where } \gamma = 1 - \frac{d}{S}, \quad C = C(\gamma),$$

which completes the proof.  $\square$

4. By Lemma 6,  $|\Pi_1(z)/\Pi_1(0)| = 1 + O((\log n)/n)$ . On the other hand, applying Lemma 7 to  $\Pi_1(0)/h_1(x^*, 0)$  we obtain (since  $S = sd$ )

$$\Pi_1(0) \leq Cn^{1-1/s}h_1(x^*, 0).$$

Together with the estimate (2.14) for  $\Pi_2$ , this leads to

$$|h_1(x, z)|/h_1(x^*, 0) \leq Cn^{1-1/s}(1/n)^{(s+r)p}.$$

Let us take into account the coefficients  $a, b$  of the potential. We set

$$(2.19) \quad T = \max\{|a|, |b|\};$$

then

$$(2.20) \quad \frac{|h(x, z)|}{|h(x^*, 0)|} = \frac{|a|^{\tilde{p}}|b|^{\tilde{q}}|h_1(x, z)|}{|b|^{rm}h_1(x^*, 0)} \leq Cn^{1-1/s} \left(\frac{T}{n}\right)^{(s+r)p},$$

because  $\tilde{p} + \tilde{q} - rm = (r+s)p$  due to (2.4) and (2.7).

5. The number of paths  $x \in X_n(p)$  does not exceed

$$(2.21) \quad \#X_n(p) \leq \binom{\tilde{p} + \tilde{q}}{\tilde{p}}.$$

In view of (2.7),

$$(2.22) \quad \#X_n(p) \leq \binom{(s+r)p + rm}{sp} \leq \begin{cases} \frac{1}{(sp)!} [(s+2r)m]^{sp} & \text{if } p \leq m, \\ 2^{(s+2r)p} & \text{if } p > m. \end{cases}$$

By (2.20) and (2.22), it follows that

$$(2.23) \quad \sum_{p=1}^{\infty} \sum_{x \in X_n(p)} |h(x, z)| \leq |h(x^*, 0)|(\sigma_1 + \sigma_2),$$

where

$$\begin{aligned} \sigma_1 &= Cn^{1-\frac{1}{s}} \sum_{p=1}^m \frac{1}{(sp)!} [(s+2r)m]^{sp} \left(\frac{T}{n}\right)^{(s+r)p}, \\ \sigma_2 &= Cn^{1-\frac{1}{s}} \sum_{p=m+1}^{\infty} 2^{(s+2r)p} \left(\frac{T}{n}\right)^{(s+r)p}. \end{aligned}$$

Since  $n = rsdm$  we have

$$[(s+2r)m]^{sp} \left(\frac{T}{n}\right)^{(s+r)p} = \left(T \frac{s+2r}{rsd}\right)^{sp} \left(\frac{T}{n}\right)^{rp} = \left(\frac{T_1}{n}\right)^{rp}$$

where  $T_1 = T \left(T \frac{s+2r}{rsd}\right)^{s/r}$ . Therefore, for  $n \geq 2T_1 + 1$ ,

$$\sigma_1 = Cn^{1-\frac{1}{s}} \sum_{p=1}^m (T_1/n)^{rp} \leq 2Cn^{1-\frac{1}{s}} (T_1/n)^r \leq C_1 n^{1-r-\frac{1}{s}},$$

where  $C_1 = C_1(r, s, T)$ .

The second sum  $\sigma_2$  is much smaller than the first one:

$$\sigma_2 \leq Cn^{1-1/s} \sum_{p=m+1}^{\infty} \left(\frac{4T}{n}\right)^{(s+r)p} \leq C_2 n^{1-(s+r)(m+1)-\frac{1}{s}},$$

where  $C_2 = C_2(r, s, T)$ .

In view of (2.23), the obtained estimates for  $\sigma_1$  and  $\sigma_2$  prove that

$$(2.24) \quad \sum_{p=1}^{\infty} \sum_{x \in X(p)} |h(x, z)| \leq C(r, s, T) |h(x^*, 0)| n^{1-r-\frac{1}{s}}, \quad |z| \leq 1.$$

Hence, the following is true.

**Lemma 8.** *For large enough  $n = mdsr$ ,  $m \in \mathbb{N}$ ,*

$$(2.25) \quad \frac{1}{2} \beta_n^+(0) \leq |\beta_n^+(z)| \leq 2\beta_n^+(0)$$

and

$$(2.26) \quad \beta_n^+(0) = h(x^*, 0) \left( 1 + O\left(n^{1-r-\frac{1}{s}}\right) \right)$$

with

$$(2.27) \quad h(x^*, 0) = 4s^2 d^2 \left( \frac{b}{4s^2 d^2} \right)^{rm} ((rm - 1)!)^{-2}.$$

6. To analyze the paths  $y \in Y_n$  from  $n$  to  $-n$ , i.e.,

$$(2.28) \quad \sum_1^{\nu+1} y(t) = -2n,$$

we can just exchange the roles of  $R$  and  $S$  and repeat the above statements with proper adjustments. Then

$$Y_n(0) = \{y^*\}, \quad y^*(t) = -2R, \quad 1 \leq t \leq sm - 1,$$

and the following holds.

**Lemma 9.** *For large enough  $n = mdsr$ ,  $m \in \mathbb{N}$ ,*

$$(2.29) \quad \frac{1}{2}\beta_n^-(0) \leq |\beta_n^-(z)| \leq 2\beta_n^-(0)$$

and

$$(2.30) \quad \beta_n^-(0) = h(y^*, 0) \left( 1 + O\left(n^{1-s-\frac{1}{r}}\right) \right)$$

with

$$(2.31) \quad h(y^*, 0) = 4r^2 d^2 \left( \frac{a}{4r^2 d^2} \right)^{sm} ((sm - 1)!)^{-2}.$$

**Remark 10.** *If  $R = 1$  then  $d = r = 1$ ,  $S = s$ , and for any  $n$  if we go backward from  $+n$  to  $-n$  it could be done without using forward steps  $+2s$ . Analogues of (2.30) could be given for any  $s$  – see Section 3.5.*

7. The set  $\Delta$  defined in (2.5) certainly contains infinitely many even integers because  $m$  could run over  $2\mathbb{N}$ . But if  $rsd$  is even, then  $\Delta \cap (2\mathbb{N} + 1) = \emptyset$  while  $\Delta \cap (2\mathbb{N} + 1)$  is infinite if  $rsd$  is odd, i.e., if  $R$  and  $S$  are odd. In any case, if  $R \neq S$ , say  $R < S$ ,

$$\min\{|\beta_n^\pm(0)/\beta_n^\mp(0)|, n = (rsd)m\} \leq (C_3)^m \frac{(rm - 1)!}{(sm - 1)!} \leq (C_4)^m m^{-|r-s|m}.$$

In view of Criterion 3, these observations lead to the following.

**Theorem 11.** *For any potential  $v$  in (2.1) there is no basis consisting of root functions of  $L_{Per^+}(v)$ . If  $R$  and  $S$  are odd, the same is true for  $L_{Per^-}(v)$ .*

3. POTENTIALS  $ae^{-2ix} + be^{2s ix}$ ,  $s > 2$ .

1. If we analyze  $bc = Per^-$  in the case the potential is of the form (2.1) and one of the parameters  $r, s$  in (2.2) is even then the constructions in Section 2 cannot be applied to give us a negative statement like Theorem 11. In this section we present elaborate analysis in the case  $r = 1$ ,  $s > 2$  and

$$(3.1) \quad \Delta = \{n = sm - 1, m \in \mathbb{N}\}.$$

Observe, that if  $s$  is even, then  $\Delta$  consist of odd numbers, and if  $s$  is odd then  $\Delta \cap (2\mathbb{N} - 1) \neq \emptyset$  and  $\Delta \cap 2\mathbb{N} \neq \emptyset$ . So, by showing that

$$\inf\{|\beta_n^\pm(0)|/|\beta_n^\mp(0)| : n \in \Delta, n \geq N(v)\} = 0$$

we would obtain by Criterion 3 that there is *no* basis in  $L^2([0, \pi])$  consisting of root functions of  $L_{Per^-}(v)$  for potentials of the form

$$(3.2) \quad v(x) = ae^{-2ix} + Be^{2s ix}, \quad a, b \neq 0, \quad s \geq 3.$$

Let us remind that Theorem 11 in Section 2 considers the operators  $L_{Per^+}(v)$  for any  $s$ . Its claim follows from Criterion 3 because

$$\inf\{|\beta_n^\pm(0)|/|\beta_n^\mp(0)| : n \in s\mathbb{N}, n \geq N(v)\} = 0.$$

In the sequel we write for convenience  $h_1(x)$  instead of  $h_1(x, 0)$ , and  $h(x)$  instead of  $h(x, 0)$ .

2. Fix  $n = sm - 1$ ; a path  $x = (x(t))_{t=1}^{\nu+1}$  from  $-n$  to  $n$  gives a non-zero term  $h(x, z)$  in  $\beta_n^+(z)$  if and only if (compare with (2.3))

$$(3.3) \quad x(t) = -2 \quad \text{or} \quad x(t) = 2s.$$

Set

$$(3.4) \quad \begin{aligned} p &= \#\{t : x(t) = -2, \quad 1 \leq t \leq \nu(x) + 1\}, \\ q &= \#\{t : x(t) = 2s, \quad 1 \leq t \leq \nu(x) + 1\}; \end{aligned}$$

then we have

$$(3.5) \quad 2n = -2p + 2sq \Rightarrow sm - 1 = -p + sq \Rightarrow p = 1 + s(q - m).$$

We set

$$(3.6) \quad p = 1 + s\kappa, \quad q = m + \kappa$$

to satisfy (3.5), and define  $X_n(\kappa)$  as the set of all admissible paths satisfying (3.3) which parameters  $p$  and  $q$  are given by (3.6). Then

$$(3.7) \quad \#X_n(0) = m + 1,$$

and with  $p = 1$ ,  $q = m$  a path  $\xi^\tau \in X_n(0)$  is uniquely determined by the position  $\tau$  of its only step  $-2$ . In other words, the paths in  $X_n(0)$  are given by

$$(3.8) \quad \xi^\tau(t) = \begin{cases} 2s, & t \neq \tau \\ -2, & t = \tau \end{cases} \quad 1 \leq \tau, t \leq m+1.$$

Among them the two paths  $\xi^1$  and  $\xi^{m+1}$  are special in the sense that  $h_1(\xi^1) = h_1(\xi^{m+1}) < 0$ , while  $h_1(\xi^\tau) > 0$  for  $\tau = 2, \dots, m$ . More precisely, since  $j(t, \xi^1) = -n - 2 + 2s(t - 1)$ , we have

$$n^2 - j(1, \xi^1)^2 = n^2 - (-n - 2)^2 = -4(n + 1) = -4ms$$

and

$$n^2 - j(t+1, \xi^1)^2 = n^2 - (-n - 2 + 2st)^2 = 4s(m-t)(st-1), \quad t = 1, \dots, m-1,$$

so it follows that

$$(3.9) \quad h_1(\xi^1) = \prod_{t=1}^m [n^2 - j(t, \xi^1)^2]^{-1} = \frac{-1}{(4s)^m m!} \left( \prod_{t=1}^{m-1} (st - 1) \right)^{-1}.$$

By symmetry  $h_1(\xi^{m+1}) = h_1(\xi^1)$ , so we obtain for their sum

$$(3.10) \quad h_1(\xi^1) + h_1(\xi^{m+1}) = -H^-(m)$$

with

$$(3.11) \quad H^-(m) = \frac{2}{(4s)^m m!} \left( \prod_{t=1}^{m-1} (st - 1) \right)^{-1} = \frac{2s}{(2s)^{2m} m!} \frac{\Gamma(1 - \frac{1}{s})}{\Gamma(m - \frac{1}{s})}.$$

For  $\xi^\tau$  with  $2 \leq \tau \leq m$  we have

$$(3.12) \quad j(t, \xi^\tau) = \begin{cases} -n + 2st, & t \leq \tau - 1, \\ -n - 2 + 2s(t - 1), & \tau \leq t \leq m. \end{cases}$$

By (3.1) and (3.12)

$$n^2 - j(\xi^\tau, t)^2 = \begin{cases} 4st[(m-t)s - 1], & 1 \leq t \leq \tau - 1, \\ 4s(s(t-1) - 1)(m - (t-1)), & \tau \leq t \leq m, \end{cases}$$

which implies, for  $2 \leq \tau \leq m$ , that

$$(3.13) \quad h_1(\xi^\tau) = \frac{1}{(4s)^m (\tau - 1)! (m - \tau + 1)!} \left( \prod_{t=m-\tau+1}^{m-1} (st - 1) \right)^{-1} \left( \prod_{t=\tau-1}^{m-1} (st - 1) \right)^{-1}.$$

One can easily see that the sum

$$(3.14) \quad H^+ = H^+(m) := \sum_2^m h_1(\xi^\tau)$$

can be written (if we change  $\tau$  to  $\tau - 1$ ) as

$$(3.15) \quad H^+ = \frac{1}{(4s)^m} \sum_{\tau=1}^{m-1} \frac{\prod_1^{\tau-1}(st-1)}{\tau!} \frac{\prod_1^{m-\tau-1}(st-1)}{(m-\tau)!} \left( \prod_{t=1}^{m-1} (st-1) \right)^{-2}.$$

We set  $\alpha = 1/s$ ; then

$$(3.16) \quad \alpha < 1/2 \quad (\text{so } 1 - 2\alpha > 0) \quad \text{for } s > 2.$$

Let

$$(3.17) \quad A_\alpha(k) = \frac{\alpha \prod_1^{k-1}(t-\alpha)}{k!} = \frac{\alpha \Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}, \quad k \geq 2,$$

$$(3.18) \quad A_\alpha(0) = 0, \quad A_\alpha(1) = \alpha.$$

Then

$$(3.19) \quad 2A_\alpha(m) \times (H^+/H^-) = \sum_{\tau=1}^{m-1} A_\alpha(\tau) A_\alpha(m-\tau),$$

and

$$(3.20) \quad \sum_{k=0}^{\infty} A_\alpha(k) w^k = f_\alpha(w) := 1 - (1-w)^\alpha$$

happens to be a nice generating function. The right-hand side of (3.19) is the  $m$ -th Taylor coefficient  $T_m$  of the square

$$(f_\alpha(w))^2 = (1 - (1-w)^\alpha)^2 = 1 - 2(1-w)^\alpha + (1-w)^{2\alpha} = 2f_\alpha(w) - f_{2\alpha}(w),$$

so it equals

$$T_m([f_\alpha]^2) = 2A_\alpha(m) - A_{2\alpha}(m)$$

Hence, dividing by  $2A_\alpha(m)$  and taking into account (3.16) and (3.17), we obtain

$$(3.21) \quad \frac{H^+}{H^-} = 1 - \frac{A_{2\alpha}(m)}{2A_\alpha(m)} = 1 - \frac{\Gamma(1-\alpha)\Gamma(m-2\alpha)}{\Gamma(m-\alpha)\Gamma(1-2\alpha)}, \quad \alpha = 1/s.$$

The Stirling formula shows that

$$(3.22) \quad r(m) := \frac{A_{2\alpha}(m)}{2A_\alpha(m)} = \frac{\Gamma(1-\alpha)\Gamma(m-2\alpha)}{\Gamma(m-\alpha)\Gamma(1-2\alpha)} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \rho(m) m^{-\alpha},$$

where  $\rho(m) \rightarrow 1$ . Therefore,

$$(3.23) \quad \frac{H^- - H^+}{H^- + H^+} = \frac{r(m)}{2 - r(m)} \asymp \frac{1}{2}r(m)$$

for large enough  $m$ , i.e., we proved the following.

**Lemma 12.** *In the above notations,*

$$(3.24) \quad H^-(m) - H^+(m) \gtrsim m^{-1/s} (H^-(m) + H^+(m)) \quad \text{as } m \rightarrow \infty.$$

By Lemma 6, for large enough  $n$  and  $|z| \leq 1$  we have that

$$(3.25) \quad h_1(\xi, z) = h_1(\xi, 0)(1 + \theta(\xi, z)), \quad |\theta(\xi, z)| \leq \frac{4 \log n}{n}, \quad \xi \in X_n(0).$$

Indeed, if  $\xi = \xi^\tau$ ,  $\tau = 2, \dots, m-1$  then (3.25) follows directly from Lemma 6. To handle  $h_1(\xi^1, z)$ , we write it in the form

$$h_1(\xi^1, z) = \frac{1}{n^2 - (-n-2)^2 + z} \prod_{k=1}^{m-1} \frac{1}{n^2 - (-n-2+2sk)^2 + z}.$$

Then we apply Lemma 6) to the product on the right and estimate the single factor by  $(-4n-4+z)^{-1} = -(4n+4)^{-1}(1 + O(1/n))$ . The case  $\xi = \xi^{m+1}$  is symmetric.

From (3.25) and (3.10) it follows that

$$(3.26) \quad h_1(\xi^1, z) + h_1(\xi^{m+1}, z) = -H^-(m) [1 + O((\log n)/n)].$$

On the other hand, by (3.14) and (3.25) we obtain that

$$\sum_{\tau=2}^m h_1(\xi^\tau, z) = \sum_{\tau=2}^m h_1(\xi^\tau)(1 + \theta(\xi^\tau, z)) = H^+(m) + \Omega,$$

where

$$|\Omega| = \left| \sum_{\tau=2}^m h_1(\xi^\tau) \theta(\xi^\tau, z) \right| \leq \sum_{\tau=2}^m h_1(\xi^\tau) \frac{4 \log n}{n} = H^+(m) \frac{4 \log n}{n}.$$

Thus, we have

$$(3.27) \quad \sum_{\tau=2}^m h_1(\xi^\tau, z) = H^+(m)[1 + O((\log n)/n)].$$

Now (3.26) and (3.27) give us, for  $|z| \leq 1$ , that

$$(3.28) \quad \sum_{\xi \in X_n(0)} h_1(\xi^\tau, z) = (H^+(m) - H^-(m)) [1 + O((\log n)/n)].$$



3. Next we estimate the ratio of

$$\sum_{x \in X_n(\kappa)} |h_1(x, z)| \quad \text{and} \quad \sum_{\xi \in X_n(0)} |h_1(\xi)| = H^- + H^+.$$

Fix  $x \in X_n(\kappa)$ ,  $\kappa \geq 1$ , and set

$$(3.29) \quad \tau = \min[m + 1, \min\{t : x(t) = -2\}].$$

Let

$$(3.30) \quad j_k^* = j(k, \xi^\tau), \quad k = 1, \dots, m.$$

denote the vertices of  $\xi^\tau$ .

Next we choose  $m$  vertices  $j_k = j(t_k, x)$  of  $x$  so that  $j_k$  is "close" to  $j_k^*$  as follows. If  $\tau = m$  or  $\tau = m + 1$  we set  $j_k = j_k^*$ ,  $k = 1, \dots, m$ . If  $\tau < m$  we set

$$(3.31) \quad t_k = k \quad \text{if } 1 \leq k \leq \tau$$

and

$$(3.32) \quad t_k = \min\{t > \tau : j(t, x) > j_{k-1}^*\}, \quad \tau + 1 \leq k \leq m.$$

Let  $J(x) := (j(t, x))_{t=1}^{\nu(x)}$  be the sequence of the vertices of  $x$ . The sequence  $(j_k)_{k=1}^m = (j(t_k, x))_{k=1}^m$  is a subsequence of  $J(x)$ ; let

$$I(x) = (i_1, \dots, i_\rho), \quad \rho = \nu(x) - m = (1 + s)\kappa$$

be its complementary subsequence in  $J(x)$ . Consider the mapping

$$(3.33) \quad \Phi_\kappa : X_n(\kappa) \rightarrow X_n(0) \times \mathbb{Z}^{(1+s)\kappa}, \quad \Phi_\kappa(x) = (\xi^\tau, I(x)).$$

**Lemma 13.** *The mapping  $\Phi_\kappa$  is injective.*

*Proof.* The lemma will be proved if we show that given  $\Phi_\kappa(x) = (\xi^\tau, I(x))$  we can restore in a unique way the path  $x$  (or equivalently, the sequence of its vertices  $J(x)$ ).

In view of the construction, if  $\tau = m$  or  $\tau = m + 1$  then

$$J(x) = (j_1^*, \dots, j_m^*, i_1, \dots, i_\rho), \quad \rho = \nu(x) - m.$$

In the case  $\tau < m$  we have to find the vertices  $j_k$  and their places in  $J(x)$ . By (3.31),

$$j_k = j(k, x) = j_k^*, \quad 1 \leq k \leq \tau.$$

Consider the first term  $i_1$  of the sequence  $I(x)$ . By (3.32), there is an integer  $\mu_1$  such that  $0 \leq \mu_1 \leq m - \tau$  and

$$i_1 - j_\tau^* = -2 + 2s \cdot \mu_1;$$

then  $j_k = j(k, x) = j_\tau^* + 2s(k - \tau)$  for  $\tau + 1 \leq k \leq k_1 := \tau + \mu_1$ . If  $k_1 = m$  we have  $j(t, x) = i_{t-m}$  for  $m + 1 \leq t \leq \nu(x)$ , so  $J(x)$  is restored.

Otherwise, we set

$$\tau_1 = \min\{t : i_{t+1} - i_t \notin \{-2, 2s\}, 1 \leq t < \rho\}.$$

From (3.32) it follows that  $i_1, \dots, i_{\tau_1}$  are successive vertices of  $x$ , so we have

$$j(t, x) = i_{t-k_1}, \quad k_1 + 1 \leq t \leq \tau_1 + k_1.$$

Moreover, there is  $\mu_2 \in \mathbb{N}$  such that

$$i_{\tau_1+1} - i_{\tau_1} = -2 + 2s \cdot \mu_2,$$

which implies

$$j_k = i_{\tau_1} + 2s(k - k_1), \quad k_1 + 1 \leq k \leq k_2 := k_1 + \mu_2,$$

so

$$j(t, x) = i_{\tau_1} + 2s(t - \tau_1 - k_1), \quad \tau_1 + k_1 + 1 \leq t \leq \tau_1 + k_2.$$

In the case  $k_2 = m$  we have  $j(t, x) = i_{t-m}$  for  $m + \tau_1 + 1 \leq t \leq \nu(x)$ , so  $J(x)$  is restored. Otherwise, we set

$$\tau_2 = \min\{t : i_{t+1} - i_t \notin \{-2, 2s\}, \tau_1 + 1 \leq t < \rho\}.$$

and continue by induction.  $\square$

Fix  $x \in X_n(\kappa)$ , and let  $(j_k)_{k=1}^m$  and  $\Phi(x) = (\xi^\tau, I(x))$  be defined as above. Then

$$(3.34) \quad h_1(x, z) = \prod_{k=1}^m (n^2 - j_k^2 + z)^{-1} \cdot \prod_{i \in I(x)} (n^2 - i^2 + z)^{-1},$$

and by Lemma 6 we have

$$\prod_{k=1}^m (n^2 - j_k^2 + z)^{-1} = \left( \prod_{k=1}^m (n^2 - j_k^2)^{-1} \right) (1 + O((\log n)/n)).$$

On the other hand, by (3.31) and (3.32),  $j_k = j_k^*$  for  $1 \leq k \leq \tau$  and  $j_{k-1}^* < j_k \leq j_k^*$  for  $\tau < k \leq m$ . Therefore, by Lemma 7 we obtain

$$\frac{1}{h_1(\xi^\tau)} \prod_{k=1}^m (n^2 - j_k^2)^{-1} = \prod_{k=\tau}^m \frac{n^2 - (j_k^*)^2}{n^2 - (j_k)^2} \leq C n^{1-\frac{1}{s}}.$$

(Since  $j_k^* = n - 2s(m + 1 - k)$ , we apply Lemma 7 after changing the summation index by  $\tilde{k} = m + 1 - k$ .) Thus, the above inequalities imply that

$$(3.35) \quad \prod_{k=1}^m (n^2 - j_k^2 + z)^{-1} \leq C h(\xi^\tau) n^{1-\frac{1}{s}}.$$

Let  $X_n(\kappa, \tau)$  be the set of all  $x \in X_n(\kappa)$  such that (3.29) holds. The sets  $X_n(\kappa, \tau)$ ,  $1 \leq \tau \leq m+1$  are disjoint, and

$$X_n(\kappa) = \bigcup_{\tau=1}^{m+1} X_n(\kappa, \tau).$$

In view of (3.34) and (3.35) we have

$$(3.36) \quad \sum_{x \in X_n(\kappa, \tau)} |h_1(x, z)| \leq C n^{1-\frac{1}{s}} |h_1(\xi^\tau)| \sum_{x \in X_n(\kappa, \tau)} \prod_{i \in I(x)} |n^2 - i^2 + z|^{-1}.$$

By Lemma 12 the mapping  $\Phi_\kappa$  is injective, so the sequence  $I(x) = (i_1, \dots, i_\rho)$  is uniquely determined by  $x \in X_n(\kappa, \tau)$ . Moreover,

$$|n^2 - i^2 + z| \geq |n^2 - i^2| - 1 \geq \frac{1}{2} |n^2 - i^2| \quad \text{for } |z| \leq 1.$$

Therefore,

$$\begin{aligned} \sum_{x \in X_n(\kappa, \tau)} \prod_{i \in I(x)} |n^2 - i^2 + z|^{-1} &\leq \sum_{i_1, \dots, i_\rho \neq \pm n} \frac{2^\rho}{|n^2 - i_1^2| \cdots |n^2 - i_\rho^2|} \\ &= \left( \sum_{i \neq \pm n} \frac{2}{|n^2 - i^2|} \right)^\rho \leq \left( \frac{C \log n}{n} \right)^\rho, \quad \rho = (s+1)\kappa, \end{aligned}$$

because  $\sum_{i \neq \pm n} |n^2 - i^2|^{-1} \leq (C \log n)/n$  (e.g., see Lemma 10 in [4]). Therefore, taking a sum over  $\tau = 1, \dots, m+1$  in (3.36), we obtain the following.

**Lemma 14.** *In the above notations, for  $\kappa = 1, 2, \dots$ ,*

$$(3.37) \quad \sum_{x \in X_n(\kappa)} |h_1(x, z)| \leq C n^{1-\frac{1}{s}} \left( \frac{C \log n}{n} \right)^{(s+1)\kappa} (H^- + H^+).$$

4. Now we are going to show that the main term of the asymptotics of  $\beta_n^+(z)$ ,  $|z| \leq 1$ , is given by  $H^+ - H^-$ . First we prove the following.

**Lemma 15.** *In the above notations, for  $n = sm - 1$ , we have*

$$(3.38) \quad \sum_{x \in X_n \setminus X_n(0)} |h(x, z)| \leq C(a, b) \frac{\log n}{n^{1/s}} \left( \frac{\log n}{n} \right)^s (H^-(m) + H^+(m)).$$

*Proof.* If  $x \in X_n(\kappa)$ , then  $\nu(x) = p + q$  with  $p = 1 + s\kappa$ , and  $q = m + \kappa$ , so

$$h(x, z) = a^p b^q h_1(x, z) = ab^m (a^s b)^\kappa h_1(x, z).$$

By (3.37) it follows

$$\sum_{x \in X_n(\kappa)} |h(x, z)| \leq C|a||b|^m n^{1-\frac{1}{s}} (D(n))^\kappa (H^-(m) + H^+(m)),$$

with  $D(n) = |a^s b| \left(\frac{C \log n}{n}\right)^{s+1}$ . For large enough  $n$  we have  $D(n) < 1/2$ , so

$$\sum_{\kappa=1}^{\infty} \sum_{x \in X_n(\kappa)} |h(x, z)| \leq C|a||b|^m n^{1-\frac{1}{s}} D(n) (H^-(m) + H^+(m)),$$

which completes the proof. □

Since

$$\beta_n^+(z) = \sum_{x \in X_n(0)} h(x, z) + \sum_{x \in X_n \setminus X_n(0)} h(x, z),$$

Lemma 12 and Lemma 15 lead to the following.

**Proposition 16.** *In the above notations, for  $n = sm - 1$ , we have*

$$(3.39) \quad \beta_n^+(z) = \beta_n^+(0)[1 + O((\log n)/n)],$$

where

$$(3.40) \quad \beta_n^+(0) = \frac{-2sab^m}{(2s)^{2m}m!} \frac{\Gamma^2(1 - \frac{1}{s})\Gamma(m - \frac{2}{s})}{\Gamma^2(m - \frac{1}{s})\Gamma(1 - \frac{2}{s})} (1 + O((\log n)^{s+1}/n^s)).$$

*Proof.* Indeed, by (3.24) one can easily see that (3.39) follows from (3.28) and (3.38).

To prove (3.40), let us recall that

$$\sum_{\xi \in X_n(0)} h(\xi, 0) = H^+(m) - H^-(m),$$

so (3.38) and (3.24) imply that

$$(3.41) \quad \beta_n^+(0) = ab^m (H^+(m) - H^-(m)) (1 + O((\log n)^{s+1}/n^s)).$$

Therefore, (3.40) follows from (3.11) and (3.21), which completes the proof. □

5. Next we estimate  $\beta_n^-(z)$  for  $|z| \leq 1$  – compare Lemma 8 – without any restriction like (2.5) or (3.1) on  $n$ . For every  $n$ , if  $y$  is a path from  $+n$  to  $-n$  satisfying (3.3) – (3.4), then we have  $-2n = -2p + 2sq$ , i.e.,

$$(3.42) \quad p = n + sq.$$

We define  $Y_n(q)$  as the set of all paths (3.3) with parameters  $p, q$  satisfying (3.42). Then

$$(3.43) \quad \#Y_n(0) = 1$$

and the only path  $\eta \in Y_n(0)$  is defined by

$$(3.44) \quad \eta(t) = -2, \quad 1 \leq t \leq n,$$

so its vertices are

$$(3.45) \quad j(t; \eta) = n - 2t, \quad 0 \leq t \leq n.$$

Therefore,

$$(3.46) \quad h_1(\eta) = \prod_{t=1}^{n-1} [n^2 - (n - 2t)^2]^{-1} = \frac{1}{4^{n-1}[(n-1)!]^2}$$

and, due to Lemma 6,

$$(3.47) \quad h_1(\eta, z) = \prod_{t=1}^{n-1} [n^2 - (n - 2t)^2 + z]^{-1} = h_1(\eta) [1 + O((\log n)/n)].$$

If  $q \geq 1$ , then any path  $y \in Y_n(q)$  has a sub-path with  $s + 1$  steps of the form  $(2s, -2, \dots, -2)$ . Indeed, choose

$$(3.48) \quad t^* = \max\{t : y(t) = 2s\};$$

then  $t^* \leq \nu(y) - s - 1$ , and

$$(3.49) \quad y(t) = -2, \quad t^* + 1 \leq t \leq t^* + s.$$

Now define a new path  $\tilde{y} \in Y(q - 1)$  by

$$(3.50) \quad \tilde{y}(t) = \begin{cases} y(t), & 1 \leq t < t^*, \\ y(t + 1 + s), & t^* \leq t \leq \nu(y) - s. \end{cases}$$

Then

$$(3.51) \quad h_1(y, z) = h_1(\tilde{y}, z) \cdot \prod_{t=t^*}^{t^*+s} [n^2 - (n - j^2(t, y))^2 + z]^{-1},$$

so

$$|h_1(y, z)| \leq (2n)^{-(s+1)} |h_1(\tilde{y}, z)| \quad \text{for } |z| \leq 1.$$

After  $q$  such restructuring we come, in view of (3.47), to the inequality

$$(3.52) \quad |h_1(y, z)| \leq 2(2n)^{-q(s+1)} |h_1(\eta)|, \quad |z| \leq 1, \quad n > N_1.$$

If  $T = \max\{|a|, |b|\}$ , then – compare (2.19) - (2.20) – for  $y \in Y_n(q)$  it follows from (3.52) that

$$(3.53) \quad |h(y, z)| = |a^{n+qs}b^q| |h_1(y, z)| \leq \frac{2T^{q(s+1)}}{(2n)^{q(s+1)}} |a^n h_1(\eta)| = 2|h(\eta)| \left(\frac{T}{2n}\right)^{q(s+1)}.$$

As in (2.21), now we can claim that

$$(3.54) \quad \#Y_n(q) \leq \binom{p+q}{q} = \binom{n+q(s+1)}{q} \leq \begin{cases} \frac{1}{q!}(s+2)^q n^q & \text{if } q < n, \\ 2^{(s+2)q} & \text{if } q \geq n. \end{cases}$$

Therefore, by (3.53) and (3.54) we obtain

$$(3.55) \quad \sum_{q \geq 1} \sum_{y \in Y_n(q)} |h(y, z)| \leq 2|h(\eta)|(\sigma_1 + \sigma_2),$$

where for large enough  $n$

$$(3.56) \quad \sigma_1 = \sum_{q=1}^{n-1} \frac{(s+2)^q n^q}{q!} \left(\frac{T}{2n}\right)^{q(s+1)} \leq C_1 n^{-s},$$

and

$$(3.57) \quad \sigma_2 = \sum_{q=n}^{\infty} 2^{q(s+2)} \left(\frac{T}{2n}\right)^{q(s+1)} \leq 2^q \left(\frac{T}{n}\right)^{n(s+1)}.$$

Certainly, the inequalities (3.54) - (3.57) imply

$$(3.58) \quad \sum_{Y_n \setminus \{\eta\}} |h(y, z)| \leq \frac{C}{n^s} |h(\eta)|, \quad |z| \leq 1.$$

**Proposition 17.** *In the above notations,*

$$(3.59) \quad \beta_n^-(z) = \beta_n^-(0)(1 + O((\log n)/n)),$$

where

$$(3.60) \quad \beta_n^-(0) = \frac{a^n}{4^{n-1}[(n-1)!]^2} (1 + O(1/n^s)).$$

*Proof.* Indeed, (3.60) follows from (3.46) and (3.58), and (3.59) follows from (3.46), (3.60), (3.47) and (3.58).  $\square$

**Theorem 18.** *For any potential of the form*

$$(3.61) \quad v(x) = ae^{-2ix} + be^{2isx}, \quad a, b \neq 0, \quad s \geq 3,$$

*there is no basis consisting of root functions of  $L_{Per-}(v)$ .*

*Proof.* In view of (3.39), (3.40) and (3.59), we may apply Criterion 3 to the set  $\Delta = \{n = sm - 1, m \in \mathbb{N}\}$ . By (3.40), (3.59) and the Stirling formula, we have

$$(3.62) \quad |\beta_n^-(0)|/|\beta_n^+(0)| \leq C_1^n (m!/n!)^2 \leq C_2^m m^{2(1-s)m} \rightarrow 0, \quad n \in \Delta.$$

Hence, Criterion 3 implies that there is no basis consisting of root functions of  $L_{Per^-}(v)$ .  $\square$

#### 4. COMMENTS

Theorems 11 and 18 claim divergence of spectral decompositions in the case of potentials of the form

$$(4.1) \quad v(x) = ae^{-2iRx} + be^{2iSx}$$

for many pairs  $R, S$  such that  $R \neq S$ .

If  $R = S$  the picture is much simpler; it is similar to the case  $R = S = 1$  which is analyzed in [7], see Theorem 7 in Section 3 there.

If  $R = S > 1$ , then an admissible path  $x$  from  $-n$  to  $n$  (or from  $n$  to  $-n$ ) gives a nonzero term  $h(x, z)$  of  $\beta_n^+(z)$  if and only if  $x(t) = \pm 2R$ . Let  $p$  and  $q$  be, respectively, the number of steps equal to  $-2R$  and  $2R$ . Then – compare (2.1) – (2.7) –

$$(4.2) \quad 2n = -2Rp + 2Rq = 2R(p + q),$$

so

$$(4.3) \quad \beta_n^-(z) = 0, \quad \beta_n^+(z) = 0 \quad \text{if} \quad n \not\equiv 0 \pmod{R},$$

Choose  $N$  so large that (1.6) holds and the claim of Lemma 2 is valid for  $n > N$ . Set

$$(4.4) \quad \Delta_0^\pm = \{n \in \Gamma^\pm : n > N, n \not\equiv 0 \pmod{R}\}$$

and let  $E(\Delta_0^\pm) = \text{Ran } P_{\Delta_0^\pm}$ , where  $P_\Delta$  is the projection defined by (1.8)). Then, in view of (4.3), Criterion 3 implies that  $E(\Delta_0^+)$  (respectively  $E(\Delta_0^-)$ ) has a basis consisting of periodic (antiperiodic) root functions. In particular, this holds for the set  $\Delta_0$  defined by (1.10).

On the other hand, let us consider the set

$$\Delta_1^\pm = \{n \in \Gamma^\pm, n = Rm, m \geq N\}.$$

By Criterion 3 the system of root functions of  $L_{Per^\pm}$  contains a (Riesz) basis in  $L^2([0, \pi])$  if and only if  $E(\Delta_1^\pm)$  has a basis consisting of periodic (respectively antiperiodic) root functions.

One can show using the same argument as in [7, Section 3] (see Lemmas 3 and 4, and Propositions 5 and 6 there) that if  $n \in \Delta_1^\pm$ , then

$$(4.5) \quad \beta_n^+(z) = 4R^2 \left( \frac{b}{4R^2} \right)^m \frac{1}{[(m-1)!]^2} \left( 1 + O\left( \frac{\log n}{n} \right) \right), \quad |z| \leq 1,$$

$$(4.6) \quad \beta_n^-(z) = 4R^2 \left( \frac{a}{4R^2} \right)^m \frac{1}{[(m-1)!]^2} \left( 1 + O\left( \frac{\log n}{n} \right) \right), \quad |z| \leq 1.$$

Now Criterion 3 says when  $E(\Delta_1^\pm)$  has a basis consisting of root functions, which leads to the following generalization of Theorem 7 in [7].

**Proposition 19.** *If  $R$  is even, then a root function system of the operator*

$$(4.7) \quad L = -\frac{d^2}{dx^2} + ae^{-2iRx} + be^{2iRx},$$

*considered with antiperiodic boundary conditions, contains a Riesz basis in  $L^2([0, \pi])$ .*

*If  $R$  is odd and  $L$  is considered with antiperiodic boundary conditions, or  $R$  is arbitrary and  $L$  is considered with periodic boundary conditions, then the system of root functions of the operator  $L$  contains a Riesz basis in  $L^2([0, \pi])$  if and only if*

$$(4.8) \quad |a| = |b|.$$

*Proof.* By (4.5) and (4.6) we have

$$\frac{\beta_n^-(z)}{\beta_n^+(z)} = \left( \frac{a}{b} \right)^n \left( 1 + O\left( \frac{\log n}{n} \right) \right), \quad n \in \Delta_1^\pm, \quad |z| \leq 1.$$

Then the assertion follows from the simple observation that  $\Delta_1^+ \cap 2\mathbb{N}$  is an infinite set for any  $R$  but  $\Delta_1^- \cap (2\mathbb{N} - 1) = \emptyset$  if  $R$  is even and  $\Delta_1^- \cap (2\mathbb{N} - 1)$  is infinite if  $R$  is odd.  $\square$

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